

## Wealth condensation in pareto macroeconomies

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We discuss a Pareto macroeconomy (a) in a closed system with fixed total wealth and (b) in an open system with average mean wealth, and compare our results to a similar analysis in a super-open system (c) with unbounded wealth [J.-P. Bouchaud and M. Mézard, *Physica A* **282**, 536 (2000)]. Wealth condensation takes place in the social phase for closed and open economies, while it occurs in the liberal phase for super-open economies. In the first two cases, the condensation is related to a mechanism known from the balls-in-boxes model, while in the last case, to the nonintegrable tails of the Pareto distribution. For a closed macroeconomy in the social phase, we point to the emergence of a “corruption” phenomenon: a sizeable fraction of the total wealth is always amassed by a single individual.

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Power-law distributions permeate a number of phenomena in statistical physics and critical phenomena. They are an important manifestation of scale invariance as observed in fractals, self-organized criticality, and percolating structures. Generically, they are the consequence of the central limit theorem for scale-free processes where a random Lévy walk replaces Brownian motion [2].

Power-law distributions have also been suggested to describe social and economic statistics. While the bulk of the income distribution in most societies follows a log-normal distribution, about a century ago, Pareto suggested that the wealthy are outliers. The distribution of large wealths follow a power law

$$p(w) \sim w^{-1-\alpha} \quad \text{for } w \gg w_0, \quad (1)$$

with  $\alpha$  typically between 1–2. This distribution is referred to as Pareto’s distribution [3,4]. Powerlike tails also govern the distribution of income and size of firms, and the behavior of financial time series over intermediate time horizons [5]. The scale-free character of this distribution implies that the chance of an already rich individual ( $w \gg w_0$ ) to further increase his wealth by an additional factor  $\lambda$  is  $p(\lambda w)/p(w) \approx 1/\lambda^{1+\alpha}$ , independent of his current wealth and the wealth of the less fortunate. For the rich part of the ensemble, what matters is only the index  $\alpha$  and, as we will argue, the total wealth of the society.

A social engineer may attempt to use the value of  $\alpha$  to control the likelihood of large wealths in general, for instance, by increasing the global character of trade through interest rates or decreasing it through taxation. The larger  $\alpha$ , the stronger the suppression of large wealths. One may distinguish between two separate regimes, the liberal economies with  $\alpha \leq 1$  and the social ones with  $\alpha > 1$ . As we will see, the possibilities for a condensation of wealth to occur are very different in both.

The total wealth  $W$  of an economy, or alternatively, the average wealth per individual  $W/N$ , can become important as

an upper bound on the individual wealth  $w$ . For example, it is clear that there will be no rich individuals in a uniformly poor society, even if the economy is liberal ( $\alpha \leq 1$ ). Conversely, one may ask about what happens in a rich society with a restrictive social economy ( $\alpha > 1$ ). As we will show, a Pareto macroeconomy becomes unstable in this case and favors a “corrupt” scenario where one individual amasses almost all the available wealth.

To better understand the role of the macroeconomic parameter  $W/N$ , we now define the three advertised ensembles: (a) a closed economy with a total-wealth  $W$  fixed; (b) an open economy in equilibrium with external economies where  $W$  adjusts to the equilibrium mean; (c) a super-open economy where  $W$  can grow unrestricted. From the point of view of rich individuals, the essential parameters of the respective ensembles are the number  $N$  of individuals in the society that is kept fixed in all cases and (a) the Pareto index  $\alpha$  and the average individual wealth  $w = W/N$  beyond a critical value  $w_*$  (see below); (b) the Pareto index  $\alpha$  and a stability parameter  $\mu$  (see below); (c) the Pareto index  $\alpha$  only.

The authors of [1] recently proposed a simple theoretical model of a dynamical process of wealth flows, which in equilibrium, becomes a Pareto macroeconomy for the ensemble (c). In brief, the model is given by a set of stochastic equations that describe the flow of wealth in an ensemble of  $N$  individuals. Specifically, the time evolution of each individual’s wealth  $w_i(t)$ ,  $i = 1, \dots, N$ , is assumed to be described by a linear differential equation

$$\frac{dw_i(t)}{dt} = \eta_i(t)w_i(t) + \sum_{j(\neq i)}^N J_{ij}w_j(t) - \sum_{j(\neq i)}^N J_{ji}w_i(t). \quad (2)$$

Here, trading between individuals is encoded in the buy/sell flow channels  $J_{ij}$  that describe an internal macroeconomical network. In addition, each individual is subjected to an economical background that is given by a multiplicative random source  $\eta_i(t)$ , representing the spontaneous increase or decrease of wealth related to investments, gains, and losses on

the market, etc. By construction, the equations are invariant under change in monetary unit  $w_i \rightarrow \lambda w_i$ .

In general, both  $\eta_i(t)$  and  $J_{ij}$  can be very complicated functions. Following [1], here we will discuss only the simplest case, where we assume that the  $\eta_i(t)$  are just uncorrelated random variables with a Gaussian distribution, and that all interactions between individuals are the same,  $J_{ij} = J/N$  for all  $i \neq j$  (mean field). As a result, the corresponding equilibrium probability distribution has the following large- $w$  asymptotics:

$$p(w) \sim w^{-1-\alpha}, \quad (3)$$

where  $\alpha = 1 + J/\sigma^2 > 1$  and  $\sigma^2$  is the variance of the Gaussian distribution of  $\eta(t)$ . The normalization factor, which we left out of Eq. (3) for simplicity, depends on  $\alpha$  only. For large values of  $w$ , this solution gives a power law with  $\alpha > 1$ , i.e., we are in the generic situation of a social Pareto macroeconomy. However, by modifying the mean-field assumptions—considering, for example, a nontrivial network of connections  $J_{ij}$ —one could also obtain a solution for a liberal economy,  $\alpha \leq 1$  [1].

If one calculates the average of the distribution (3), which corresponds to the average wealth of the individual, one sees that the basic difference between a social and a liberal economy is that it is finite in the former case and infinite in the latter. Thus, for  $\alpha \leq 1$  one would, due to the nonintegrable tail of the distribution, expect the appearance of a rich individual in the ensemble, with a wealth  $N^{1/\alpha}$  times larger than the typical value. The authors of [1] interpreted this result as a condensation phenomenon.

In reality, this is not the case and the total wealth of the society  $W$  is, in general, fixed, thereby upsetting overall scale invariance and giving us a closed system of type (a). How would the condensation phenomenon change in this case? One way to address this issue is to solve the equations of the type (2) on the hypersurface  $W = w_1 + \dots + w_N$ . This problem is reminiscent of Kac's master equation [6] on the sphere (fixed energy) for which a factorizable and stationary solution was derived in the thermodynamic limit under mild assumptions.

Here, we follow a more phenomenological treatment and assume that  $p(w) \sim 1/w^{1+\alpha}$  characterizes the single wealth distributions in the ensemble (a) with the individual wealths adding to  $W = w_1 + \dots + w_N$ . In this way, we have an asymptotic Pareto macroeconomy with a factorizable  $N$  distribution of wealths constrained on the hypersurface of fixed wealth  $W$ . For convenience, we assume that each individual wealth  $w_i$  is an integer given in units of the smallest available currency unit. The probability of a certain distribution of  $w_i$ 's is

$$P(w_1, \dots, w_N) = \frac{1}{Z(W, N)} \prod_i p(w_i) \delta\left(W - \sum_{i=1}^N w_i\right), \quad (4)$$

where  $Z(W, N)$  is the appropriate normalization factor,

$$Z(W, N) = \sum_{\{w_i \geq 0\}} \prod_i p(w_i) \delta\left(W - \sum_{i=1}^N w_i\right). \quad (5)$$

This model is known as the balls-in-boxes or backgammon model [7] where it has been applied to various condensation and glassy phenomena. It may be solved in the limit of an infinite number of boxes  $N$  and fixed density of balls per box  $\rho = W/N$  (thermodynamical limit) by introducing the integral representation of the delta function

$$\begin{aligned} Z(N, \rho) &= \sum_{\{w_i \geq 0\}} \prod_i p(w_i) \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda e^{-i\lambda(w_1 + \dots + w_N - \rho N)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda e^{i\lambda \rho N} \left( \sum_w p(w) e^{-i\lambda w} \right)^N \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \exp[N(i\lambda \rho + K(i\lambda))], \end{aligned} \quad (6)$$

where  $K$  is a generating function given by  $K(\sigma) = \ln \sum_{w=1}^{\infty} p(w) e^{-\sigma w}$ . Evaluating the integral using steepest descent gives

$$f(\rho) = \sigma_*(\rho) \rho + K[\sigma_*(\rho)], \quad (7)$$

where  $\sigma_*(\rho)$  is a solution of the saddle-point equation  $\rho + K'(\sigma_*) = 0$  and  $f(\rho)$  is a free-energy density per box,  $Z(W, N) = e^{Nf(\rho) + \dots}$ . For a suitable choice of the weights  $p(w) \sim 1/w^{1+\alpha}$ , the system displays a two-phase structure as the density is varied with a critical density  $\rho_{cr}$ . When  $\rho$  approaches  $\rho_{cr}$  from below,  $\sigma_*$  approaches  $\sigma_{cr}$  from above. When  $\rho$  is larger than  $\rho_{cr}$ ,  $\sigma$  becomes equal to the critical value  $\sigma_{cr}$  and the free energy is linear in  $\rho$ ,

$$f(\rho) = \sigma_{cr} \rho + \kappa_{cr}, \quad (8)$$

where  $\kappa_{cr} = K(\sigma_{cr})$ . The change of regimes at  $\rho_{cr}$  corresponds to a condensation transition, in which an extensive fraction of the balls is in a single box. The critical value  $\sigma_{cr}$  is equal to the logarithm of the radius of convergence of the series in the generating function  $K(\sigma)$ . In particular, for purely powerlike weights

$$p(w) = \frac{1}{\zeta(1+\alpha)} w^{-1-\alpha}, \quad w = 1, 2, \dots, \quad (9)$$

$\sigma_{cr} = 0$ . The normalization factor is given by the Riemann zeta function. At the end of this section, we will comment on the case when the radius of convergence of the series  $K(\sigma)$  differs from one.

The transition to a condensed phase happens when  $W/N$  becomes larger than a critical density  $w_*$ , which is nothing but the mean wealth

$$w_* = \sum_w w p(w). \quad (10)$$

Since we can change the small  $w$  part of the distribution by tuning the appropriate macroeconomical parameters without

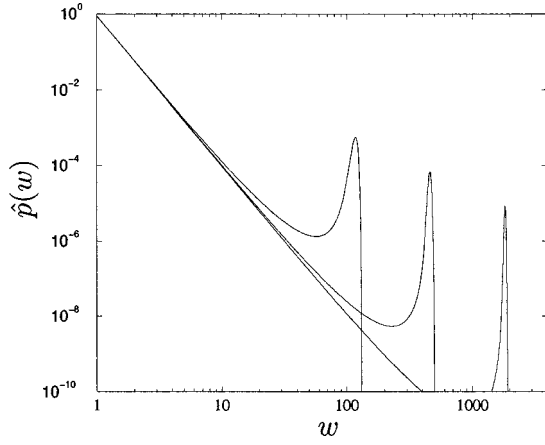


FIG. 1. The three curves, from left to right, correspond to the effective probability of a closed Pareto economy with the distribution  $p(w) = 1/w^4$  with  $N = 128, 512,$  and  $2048$  individuals, respectively, for the density  $\rho = 2$ , which lies above the critical one. The plot is done in the log-log scale. The position of the peak corresponds to the wealth of a single individual. As one can see from the figure, it grows linearly with the system size.

affecting the large  $w$  behavior of  $p(w)$ , we have some control over where the threshold  $w_*$  will lie. We can define an *effective* probability distribution of wealth

$$\hat{p}(w) = \frac{1}{N} \left\langle \sum_i^N \delta(w_i - w) \right\rangle_P, \quad (11)$$

which now, unlike the original  $p(w)$ , takes into account the finite total wealth  $W$ . Below threshold  $w_*$ , the system is in a phase in which the effective probability distribution  $\hat{p}(w)$  has an additional scale factor in comparison with the old distribution  $p(w)$

$$\hat{p}(w) \sim e^{-\sigma w} p(w). \quad (12)$$

Here,  $\sigma$  depends only on the difference  $W/N - w_*$ . It vanishes at threshold, so that the old Pareto tails are restored at this point. Above threshold, the macroeconomy responds to the increasing average wealth by creating a single individual with a wealth proportional to the total wealth  $W$ , namely,  $w_{max} = W - Nw_*$ ,

$$\hat{p}(w) \sim p(w) + \frac{1}{N} \delta_{w, W - Nw_*}. \quad (13)$$

The behavior of  $\hat{p}(w)$  versus  $w$  is shown in Fig. 1, for index  $\alpha = 3$ ,  $N = 128, 512, 2048$ , and a density  $W/N > w_*$ . At threshold, the inverse participation ratio

$$Y_2 = \frac{1}{N^2} \left\langle \sum_i w_i^2 \right\rangle_P = \frac{1}{N} \sum_w w^2 \hat{p}(w) \quad (14)$$

changes, in the large  $N$  limit, from 0 to  $(W/N - w_*)^2$ , signaling the appearance of a wealth condensation. Basically, everything in excess of the critical wealth  $Nw_*$  ends up in the portfolio of a single individual. We call this the surplus

anomaly. It can appear only in a social economy ( $\alpha > 1$ ), because only in this case do we have a finite critical wealth per individual  $w_*$ .

In a liberal economy,  $w_*$  is obviously infinite, meaning that the system remains always below threshold and there is never any condensation. Note that these results for the closed model (a) do not contradict the results of the previous section for the super-open model (c) since we now have a well-defined average wealth  $W/N$  that prevents the appearance of individuals with a wealth  $w \sim N^{1/\alpha}$  growing faster than linearly.

The behavior we have discussed here for closed systems is not restricted to power-law weights  $p(w)$ . The saddle-point equation for the generating function  $\rho + K'(\sigma_*) = 0$ , can have similar properties for other functional forms of the weights. For instance, one may easily check that a change of weights  $p(w) \rightarrow e^{-\bar{\sigma}w} p(w)$  merely leads to a change  $\sigma_{cr} \rightarrow \sigma_{cr} + \bar{\sigma}$ , leaving the phase structure of the model intact. In particular, if the weights (9) had an exponential prefactor  $p(w) \sim e^{-\bar{\sigma}w/w^{1+\alpha}}$ , we would have  $\sigma_{cr} = \bar{\sigma}$ , but the critical density

$$\rho_{cr} = \frac{\zeta(\alpha)}{\zeta(\alpha+1)} \quad (15)$$

would be independent of  $\bar{\sigma}$ . Clearly, the critical properties of the model are encoded in the subexponential behavior of the weights  $p(w)$  for large  $w$ . Solving the saddle-point equation, one may check that for weights with powerlike subexponential behavior, the most-singular part of the free-energy  $f(\rho)$  has a branch-point singularity when  $\Delta\rho = \rho_{cr} - \rho \rightarrow 0^+$

$$f(\rho) = \begin{cases} \Delta\rho^{\alpha/(\alpha-1)} & \text{for } 1 < \alpha < 2, \\ \Delta\rho^\alpha & \text{for } \alpha \geq 2. \end{cases} \quad (16)$$

For integer values, the powerlike singularity changes to a singularity of the type integer power times logarithm.

One may consider other functional subexponential forms of the weights  $p(w)$ . A criterion for the presence of the phase transition is that the derivative of the generating function is finite,  $-K'(\sigma_{cr}) < \infty$ , at the radius of convergence  $\sigma_{cr}$ . Physically, this means that the critical density is finite. For example, stretched exponential weights

$$p(w) \sim e^{-\beta w^\delta}, \quad (17)$$

with  $0 < \delta < 1$  and  $\beta > 0$  have this property. As before, we have a saddle-point phase for small density  $W/N$ , with an exponential suppression of large wealths, and a condensed phase for large density  $W/N$ , with a surplus anomaly. At the transition point, however, instead of the Pareto distribution we have Eq. (17). The second derivative of the free energy is discontinuous at the transition. If the transition is approached from the condensed phase  $\Delta\rho = \rho_{cr} - \rho \rightarrow 0^-$ ,  $f''(\rho) = 0$ , while from the saddle point one  $\Delta\rho \rightarrow 0^+$

$$f''(\rho) = \sigma_*'(\rho) = -\frac{1}{K''(\sigma_{cr})}. \quad (18)$$

For the weights (17) as well as the powerlike weights for  $\alpha > 2$ , the derivative  $K''(\sigma_{cr})$  is finite. Thus, in both cases, the second derivative of the free energy is discontinuous. In contrast, for  $1 < \alpha < 2$ ,  $K''(\sigma_{cr}) = \infty$  and  $f''(\rho) = 0$  on both sides of the transition. In this case, the singularity yielding the discontinuity of derivatives of the free energy is given by Eq. (16). The transition becomes arbitrarily soft when  $\alpha \rightarrow 1$ .

Finally, let us discuss an economy in contact with one or more external ones [ensemble (b)]. The total wealth  $W$  is not fixed in this case, but may adjust dynamically to an equilibrium value given by a stability parameter  $\mu$  (inverse temperature). The partition function for this ensemble is given by [8]

$$\mathcal{Z}(\mu, N) = \sum_W \mathcal{Z}(W, N) e^{-\mu W}. \quad (19)$$

The total wealth in our economy now depends on the value of  $\mu$ . The model has a phase transition at  $\mu = 0$ . For  $\mu > 0$ , the average wealth per individual  $W/N$  fluctuates according to a Gaussian distribution with a certain average value  $w_*(\mu)$  and a width that is inversely proportional to the square root of the system size  $1/\sqrt{N}$ . At the critical point  $\mu = 0$ , the situation becomes unstable as the economy starts to attract the attention of the outside world and  $W$  acquires a tendency to grow. In an idealized situation where the outside world has limitless wealth,  $W/N$  would actually become infinite as soon as  $\mu < 0$ . In practice, of course, it remains bounded by an upper limit.

The order parameter for this transition is  $r = N/W$ , which in the idealized case is zero for  $\mu < 0$  and positive otherwise. Its critical behavior depends on  $\alpha$  as

$$r \sim \mu^{1/\alpha} \quad \text{for } \mu \rightarrow 0^+. \quad (20)$$

The order of the transition thus depends on the type of our economy. In a social economy ( $\alpha > 1$ ), the transition is of first order and  $r$  changes discontinuously at the critical point. In a liberal economy ( $\alpha \leq 1$ ), the transition is continuous, and becomes arbitrarily soft as  $\alpha$  approaches zero.

The  $r=0$  phase is one where condensation takes place not only within the considered economy, but in the whole system including the outside world. To better illustrate this situation, consider two mean-field Pareto economies, each with the same distribution  $p(w)$  but possibly different total wealths  $W_1$  and  $W_2$ . If we bring them into contact with each other, they will form a larger mean-field economy with a constrained total wealth  $W = W_1 + W_2$ . For  $\mu = 0$ , condensation can take place with equal probability in either one of them, so if we look at only one of the systems, we might observe condensation or we might not. In other words, there are large fluctuations. But if  $\mu \neq 0$ , then one of the subsystems will favor condensation, and wealth will tend to flow towards it. The other system then has to adjust to the fact that wealth disappears from it. This leads exactly to the two phases discussed above.

We have shown that in a social economy, condensation may occur if the total wealth of the society exceeds a certain critical value. In our analysis, the system favors the occurrence of a single individual in possession of a finite fraction of the economy's total available wealth, providing a physical mechanism for "corruption." The analysis we have provided may be improved by considering Eq. (2) in general, using a random network for the flow channels restricted to a hypersurface of fixed wealth. In this way, we could learn more about the statistical aspects underlying the process of fortune creation and propagation. Clearly, these issues are also relevant to a number of physical phenomena ranging from condensation in glassy systems to symplial gravity.

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